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Classification of ruled surfaces in Minkowski 3-spaces

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Abstract

In this paper, we study some properties about the second Gaussian curvature of ruled surfaces in a three-dimensional Minkowski space. Furthermore, we classify ruled surfaces in a three-dimensional Minkowski space in terms of the second Gaussian curvature, the mean curvature and the Gaussian curvature.
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1. Introduction

Minimal surfaces are one of main objects which have drawn geometers' interest for a long time. In particular, the only minimal ruled surfaces in a three-dimensional Euclidean space $\mathbb{E}^3$ are the planes and the helicoids. In 1983, Kobayashi [8] classified space-like ruled minimal surfaces in a three-dimensional Minkowski space $\mathbb{E}^3_1$, and de Woestijne [12] extended it to the Lorentz version in 1988. On the other hand, the authors [7] recently classified minimal ruled surfaces in terms of pointwise 1-type Gauss map in $\mathbb{E}^3_1$.

A surface M in a three-dimensional Euclidean space $\mathbb{E}^3$ with positive Gaussian curvature K possesses a positive definite second fundamental form II if appropriately orientated.

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Therefore, the second fundamental form defines a new Riemannian metric on M. In turn, we can consider the Gaussian curvature K_{II} of the second fundamental form which is regarded as a Riemannian metric. If a surface has non-zero Gaussian curvature everywhere, K_{II} can be defined formally and it is the curvature of the Riemannian or pseudo-Riemannian manifold (M, II).

Naturally, we can extend such a notion into that of surfaces in a three-dimensional Minkowski space $\mathbb{E}^3_1$. Using classical notation, we denote the component functions of the second fundamental form by e, f and g. Thus we define the *second Gaussian curvature* by (cf. [2])

$$K_{II} = \frac{1}{(|eg| - f^2)^2}\left(\begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix}\right). \tag{1.1}$$

It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal.

For study of the second Gaussian curvature, Koutroufiotis [10] has shown that a closed ovaloid is a sphere if $K_{II} = cK$ for some constant c or if $K_{II} = \sqrt{K}$. Koufogiorgos and Hasanis [9] proved that the sphere is the only closed ovaloid satisfying $K_{II} = H$, where H is the mean curvature. Also, Kühnel [11] studied surfaces of revolution satisfying $K_{II} = H$. One of the natural generalizations of surfaces of revolution is the helicoidal surfaces. Baikoussis and Koufogiorgos [1] proved that the helicoidal surfaces satisfying $K_{II} = H$ are locally characterized by constancy of the ratio of the principal curvatures. On the other hand, Blair and Koufogiorgos [2] investigated a non-developable ruled surface in $\mathbb{E}^3$ such that $aK_{II} + bH$, $2a + b \neq 0$, is a constant along each ruling. Also, they proved that a ruled surface with vanishing second Gaussian curvature is a helicoid.

In this article, we investigate and classify a non-developable ruled surface in a three-dimensional Minkowski space $\mathbb{E}^3_1$ satisfying the conditions

$$aK_{II} + bH = \text{constant}, \quad 2a - b \neq 0, \tag{1.2}$$

$$aH + bK = \text{constant}, \quad a \neq 0, \tag{1.3}$$

$$aK_{II} + bK = \text{constant}, \quad a \neq 0, \tag{1.4}$$

along each ruling.

2. Preliminaries

Let $\mathbb{E}^3_1$ be a three-dimensional Minkowski space with the scalar product of index 1 given by $\langle , \rangle = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{E}^3_1$. A vector x of $\mathbb{E}^3_1$ is said to be *space-like* if $\langle x, x\rangle > 0$ or $x = 0$, *time-like* if $\langle x, x\rangle < 0$ and *light-like or null* if $\langle x, x\rangle = 0$ and $x \neq 0$. A time-like or light-like vector in $\mathbb{E}^3_1$ is said to be *causal*.

Now, we define a ruled surface M in a three-dimensional Minkowski space $\mathbb{E}^3_1$. Let J_1 be an open interval in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve in $\mathbb{E}^3_1$ defined on J_1 and $\beta = \beta(s)$ a transversal vector field along α. For an open interval J_2 of $\mathbb{R}$ we have the parametrization for M

$$x = x(s, t) = \alpha(s) + t\beta(s), \quad s \in J_1, \ t \in J_2.$$

The curve $\alpha = \alpha(s)$ is called a *base curve* and $\beta = \beta(s)$ a *director curve*. In particular, if β is constant, the ruled surface is said to be *cylindrical*, and *non-cylindrical* otherwise.

First of all, we consider that the base curve α is space-like or time-like. In this case, the director curve β can be naturally chosen so that it is orthogonal to α. Furthermore, we have ruled surfaces of five different kinds according to the character of the base curve α and the director curve β as follows. If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. In the case that β is space-like, it is said to be of type M^1_+ or M^2_+ if β' is non-null or light-like, respectively. When β is time-like, β' must be space-like by causal character. In this case, M is said to be of type M^3_+. On the other hand, for the ruled surface of type M_-, it is also said to be of type M^1_- or M^2_- if β' is non-null or light-like, respectively. Note that in the case of type M_- the director curve β is always space-like (cf. [5,7]). The ruled surface of type M^1_+ or M^2_+ (resp. M^3_+, M^1_- or M^2_-) is clearly space-like (resp. time-like). But, if the base curve α is a light-like curve and the vector field β along α is a light-like vector field, then the ruled surface M is called a *null scroll* [6].

On the other hand, many geometers have been interested in studying submanifolds of Euclidean and pseudo-Euclidean space in terms of the so-called finite type immersion [3]. Also, such a notion can be extended to smooth maps on submanifolds, namely the Gauss map [4]. In this regards, the authors defined pointwise finite type Gauss map [7]. In particular, the Gauss map G on a submanifold M of a pseudo-Euclidean space $\mathbb{E}^m_s$ of index s is said to be of *pointwise* 1-*type* if $\Delta G = fG$ for some smooth function f on M where Δ denotes the Laplace operator defined on M. The authors showed that minimal non-cylindrical ruled surfaces in a three-dimensional Minkowski space have pointwise 1-type Gauss map [7]. Based on this fact, the authors proved the following theorem which will be useful to prove our theorems in this paper.

Theorem 2.1 ([7]). *Let M be a non-cylindrical ruled surface with space-like or time-like base curve in a three-dimensional Minkowski space. Then, the Gauss map is of pointwise* 1-*type if and only if M is an open part of one of the following spaces*: *the space-like or time-like helicoid of the* 1*st, the* 2*nd and the* 3*rd kind, the space-like or time-like conjugate of Enneper's surface of the* 2*nd kind*.

3. Some examples

Before going into the study about a relation between the second Gaussian curvature and the mean curvature of ruled surfaces, let us see some examples of surfaces in $\mathbb{E}^3_1$.

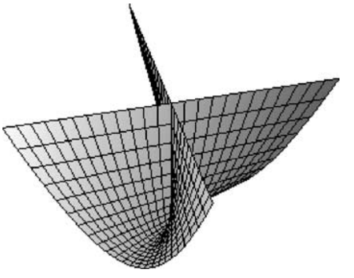

Fig. 1. A conoid of the 1st kind.

Example 3.1 (Conoid of the 1st kind). For a smooth function $\phi(s)$, we consider the surface M in $\mathbb{E}^3_1$ defined by

$$x(s,t) = (t\sinh s, t\cosh s, \phi(s)), \quad t < |\phi'(s)|.$$

This parametrization defines a non-cylindrical ruled surface of type M^1_+ in $\mathbb{E}^3_1$, which is called a conoid of the 1st kind (Fig. 1). In this case, the second Gaussian curvature K_{II} is given by

$$K_{II} = \frac{-\phi''(s)}{(-t^2+\phi'^2(s))^{3/2}}t.$$

Furthermore, the mean curvature H is obtained as

$$H = \frac{\phi''(s)}{2(-t^2+\phi'^2(s))^{3/2}}t.$$

Thus, the conoid of the 1st kind satisfies $K_{II} = -2H$.

Example 3.2 (Conoid of the 2nd kind). For a smooth function $\phi(s)$, the surface in $\mathbb{E}^3_1$ defined by

$$x(s,t) = (t\cosh s, \phi(s), t\sinh s)$$

is a non-cylindrical ruled surface of type M^3_+, which is said to be a conoid of the 2nd kind in $\mathbb{E}^3_1$ (Fig. 2). The second Gaussian curvature K_{II} is given by

$$K_{II} = \frac{\phi''(s)}{(t^2+\phi'^2(s))^{3/2}}t.$$

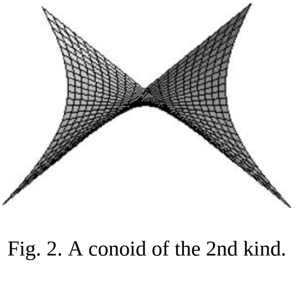

Fig. 2. A conoid of the 2nd kind.

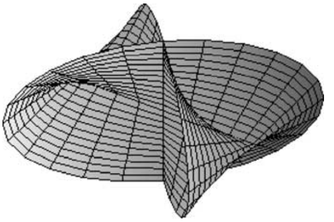

Fig. 3. A conoid of the 3rd kind.

And the mean curvature H is given by

$$H = \frac{-\phi''(s)}{2(t^2 + \phi'^2(s))^{3/2}} t.$$

Thus, the conoid of the 2nd kind satisfies $K_{II} = -2H$.

Example 3.3 (Conoid of the 3rd kind). For a smooth function $\phi(s)$, we consider the surface M in $\mathbb{E}^3_1$ defined by

$$x(s, t) = (\phi(s), t \cos s, t \sin s), \quad t < |\phi'(s)|.$$

This parametrization defines a non-cylindrical ruled surface of type M^1_- in $\mathbb{E}^3_1$, which is called a conoid of the 3rd kind (Fig. 3). The second Gaussian curvature K_{II} for it is given by

$$K_{II} = \frac{\phi''(s)}{(-t^2 + \phi'^2(s))^{3/2}} t,$$

and the mean curvature H is obtained as

$$H = \frac{-\phi''(s)}{2(-t^2 + \phi'^2(s))^{3/2}} t.$$

Thus, the conoid of the 3rd kind satisfies $K_{II} = -2H$.

For a specific function ϕ and an appropriate interval of t in Examples 3.1–3.3, we have the graphs shown in Figs. 1–3.

4. Main theorems

In this section we study a ruled surface in a three-dimensional Minkowski space $\mathbb{E}^3_1$ satisfying the conditions (1.2)–(1.4). It is well known that a cylindrical ruled surface is developable, i.e., the Gaussian curvature K is identically zero. Therefore, the second fundamental form II is degenerate. Thus, non-cylindrical ruled surfaces are meaningful for our study.

Theorem 4.1. *Let M be a non-developable ruled surface with non-null base curve in a three-dimensional Minkowski space such that the condition $aK_{II} + bH$, $a, b \in \mathbb{R} - \{0\}$,*

$2a-b\neq 0$, is a constant along each ruling. Then, M is an open part of one of the following surfaces:

1. *the helicoid of the* 1*st kind as space-like or time-like surface,*
2. *the helicoid of the* 2*nd kind as space-like or time-like surface,*
3. *the helicoid of the* 3*rd kind as space-like or time-like surface,*
4. *the conjugate of Enneper's surfaces of the* 2*nd kind as space-like or time-like surface.*

Proof. We consider two cases separately.

Case 1. Let M be a non-cylindrical ruled surface of the three types M^1_+, M^3_+ or M^1_-. Then the parametrization for M is given by

$$x = x(s,t) = \alpha(s) + t\beta(s),$$

such that $\langle\beta,\beta\rangle = \varepsilon_1(=\pm 1)$, $\langle\beta',\beta'\rangle = \varepsilon_2(=\pm 1)$ and $\langle\alpha',\beta'\rangle = 0$. In this case α is the striction curve of x, and the parameter is the arc-length on the (pseudo-)spherical curve β. And we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha' + t\beta'$ and $x_t = \beta$. Then, the first fundamental form of the surface is given by $E = \langle x_s, x_s\rangle = \langle\alpha',\alpha'\rangle + \varepsilon_2 t^2$, $F = \langle x_s, x_t\rangle = \langle\alpha',\beta\rangle$ and $G = \langle x_t, x_t\rangle = \varepsilon_1$. For later use, we define the smooth functions Q, J and D as follows:

$$Q = \langle\alpha', \beta\times\beta'\rangle \neq 0, \quad J = \langle\beta'', \beta'\times\beta\rangle, \quad D = \sqrt{|EG-F^2|}.$$

In terms of the orthonormal basis $\{\beta, \beta', \beta\times\beta'\}$ we obtain

$$\alpha' = \varepsilon_1 F\beta - \varepsilon_1\varepsilon_2 Q\beta\times\beta', \qquad \beta'' = \varepsilon_1\varepsilon_2(-\beta + J\beta\times\beta'), \qquad \alpha'\times\beta = \varepsilon_2 Q\beta'.$$

On the other hand, one obtains $EG-F^2 = -\varepsilon_2 Q^2 + \varepsilon_1\varepsilon_2 t^2$. And, the unit normal vector N is written as $N = 1/D(\varepsilon_2 Q\beta' - t\beta\times\beta')$. Then, the components e, f and g of the second fundamental form are expressed as

$$e = \frac{1}{D}(\varepsilon_1 Q(F-QJ) - Q't + Jt^2), \qquad f = \frac{Q}{D}\neq 0, \quad g = 0.$$

Therefore, using the data described above and (1.1), we obtain

$$\begin{aligned} K_{II} &= \frac{1}{f^4}\left(ff_t\left(f_s - \frac{1}{2}e_t\right) - f^2\left(-\frac{1}{2}e_{tt} + f_{st}\right)\right) \\ &= \frac{1}{2Q^2D^3}(Jt^4 + \varepsilon_1 Q(F-2QJ)t^2 + 2\varepsilon_1 Q^2Q't + Q^3(F+QJ)). \end{aligned} \tag{4.1}$$

Furthermore, the mean curvature H is given by

$$H = \frac{1}{2}\frac{Eg-2Ff+Ge}{|EG-F^2|} = \frac{1}{2D^3}(\varepsilon_1 Jt^2 - \varepsilon_1 Q't - Q(F+QJ)). \tag{4.2}$$

First of all, we suppose that $Q^2-\varepsilon_1 t^2 > 0$. We now differentiate K_{II} and H with respect to t, the results are

$$\begin{aligned} (K_{II})_t = \frac{1}{2Q^2D^5}(&-\varepsilon_1 Jt^5 + Q(F+2QJ)t^3 + 4Q^2Q't^2 \\ &+ \varepsilon_1 Q^3(5F-QJ)t + 2\varepsilon_1 Q^4Q'), \end{aligned} \tag{4.3}$$

$$H_t = \frac{1}{2D^5}(Jt^3 - 2Q't^2 - \varepsilon_1 Q(3F + QJ)t - \varepsilon_1 Q^2 Q'). \tag{4.4}$$

By the assumption (1.2) and the above equations we then have for the parameter t

$$a\varepsilon_1 J = 0, \qquad aQF + 2aQ^2 J + bJQ^2 = 0, \qquad Q^2 Q'(2a - b) = 0,$$
$$5\varepsilon_1 aQ^3 F - \varepsilon_1 aQ^4 J - 3\varepsilon_1 bQ^3 F - \varepsilon_1 bQ^4 J = 0, \qquad Q^4 Q'(2a - b) = 0, \tag{4.5}$$

which imply $J = F = 0$ and $(2a - b)Q' = 0$. Suppose that $2a - b \neq 0$, then $Q' = 0$. In this case the surface is minimal. Since $EG - F^2 = \varepsilon_1\varepsilon_2 t^2 - \varepsilon_2 Q^2$ and $Q^2 - \varepsilon_1 t^2 > 0$. Therefore, the surface is space-like or time-like when $\varepsilon_2 = -1$ or $\varepsilon_2 = 1$, respectively.

But, $(\varepsilon_1, \varepsilon_2) = (-1, -1)$ is impossible because of the causal character. Let $(\varepsilon_1, \varepsilon_2) = (-1, 1)$. Then M is of the type M_+^3. Thus the surface is a helicoid of the 3rd kind according to Theorem A. If $(\varepsilon_1, \varepsilon_2) = (1, \pm 1)$, then M is of the type M_+^1 or M_-^1. Hence the surface is a helicoid of the 1st or 2nd kind according to Theorem 2.1.

Next, we suppose that $Q^2 - \varepsilon_1 t^2 < 0$. In this case, we have

$$(K_{II})_t = \frac{1}{2Q^2 D^5}(\varepsilon_1 Jt^5 - Q(F + 2QJ)t^3 - 4Q^2 Q't^2 + \varepsilon_1 Q^3(-5F + QJ)t - 2\varepsilon_1 Q^4 Q'), \tag{4.6}$$

$$H_t = \frac{1}{2D^5}(-Jt^3 + 2Q't^2 - \varepsilon_1 Q(3F + QJ)t + \varepsilon_1 Q^2 Q'). \tag{4.7}$$

Thus, by the similar discussion as above we can also obtain $J = F = 0$ and $Q' = 0$ when $2a - b \neq 0$. Therefore, the surface is minimal. Since $EG - F^2 = -\varepsilon_2(Q^2 - \varepsilon_1 t^2)$ and $Q^2 - \varepsilon_1 t^2 < 0$. Consequently, M is space-like or time-like according to $\varepsilon_2 = 1$ or $\varepsilon_2 = -1$, respectively.

In this case, $\varepsilon_1 = 1$. Therefore, M is of type M_+^1 or M_-^1 depending on $\varepsilon_2 = \pm 1$. Thus, the surface is a helicoid of the 1st and 2nd kind according to Theorem 2.1.

Case 2. Let M be a non-cylindrical ruled surface of type M_+^2 or M_-^2. Then, the surface M is parametrized by

$$x(s, t) = \alpha(s) + t\beta(s),$$

such that $\langle\beta, \beta\rangle = 1$, $\langle\alpha', \beta\rangle = 0$, $\langle\beta', \beta'\rangle = 0$ and $\langle\alpha', \alpha'\rangle = \varepsilon_1(= \pm 1)$. We have put the non-zero smooth functions q and S as follows:

$$q = \|x_s\|^2 = \varepsilon\langle x_s, x_s\rangle = \varepsilon(\varepsilon_1 + 2St), \quad S = \langle\alpha', \beta'\rangle,$$

where ε denotes the sign of x_s. We note that $\beta \times \beta' = \beta'$. Then, the components of the induced pseudo-Riemannian metric on M are obtained by $E = \varepsilon q$, $F = 0$ and $G = 1$. For the moving frame $\{\alpha', \beta, \alpha' \times \beta\}$ we can calculate

$$\beta' = \varepsilon_1 S(\alpha' - \alpha' \times \beta), \qquad \alpha'' = -S\beta - \varepsilon_1 R\alpha' \times \beta, \tag{4.8}$$

where $R = \langle\alpha'', \alpha' \times \beta\rangle$. Furthermore, using (4.8) we have

$$\langle\beta'', \alpha' \times \beta\rangle = S' + \varepsilon_1 SR, \quad \langle\alpha', \beta''\rangle = S' + \varepsilon_1 SR.$$

The unit normal vector N is given by

$$N = \frac{1}{\sqrt{q}}(\alpha' \times \beta - t\beta'),$$

from which the coefficients of the second fundamental form are given by

$$e = \frac{1}{\sqrt{q}}(R + (S' + 2\varepsilon_1 SR)t), \quad f = \frac{S}{\sqrt{q}}, \quad g = 0.$$

On the other hand, the mean curvature H and the second Gaussian curvature K_{II} are obtained respectively by

$$H = \frac{1}{2q^{3/2}}(R + (S' + 2\varepsilon_1 SR)t), \tag{4.9}$$

$$K_{II} = \frac{\varepsilon_1 S'}{2Sq^{3/2}}. \tag{4.10}$$

Differentiating K_{II} and H with respect to t, we have

$$(K_{II})_t = \frac{-3}{2q^{5/2}}\varepsilon\varepsilon_1 S', \tag{4.11}$$

$$H_t = \frac{1}{2q^{5/2}}(\varepsilon\varepsilon_1 S' - \varepsilon SR - \varepsilon S(S' + 2\varepsilon_1 SR)t). \tag{4.12}$$

Suppose that the surface M satisfies the condition (1.2). Then, from (4.11) and (4.12) we obtain

$$\varepsilon S(S' + 2\varepsilon_1 SR) = 0, \qquad 3a\varepsilon\varepsilon_1 S' - b(\varepsilon\varepsilon_1 S' - \varepsilon SR) = 0. \tag{4.13}$$

Using (4.13), we have $(2a-b)R = 0$. Therefore $S' = 0$ and $R = 0$ when $2a - b \neq 0$. Thus, from (4.9) M is minimal, that is, it is a conjugate of Enneper's surface of the 2nd kind as space-like or time-like surface according to Theorem 2.1. This completes the proof. □

Remark. In Case 1 of Theorem 4.1, if $2a - b = 0$, the surface M satisfies the equation $K_{II} = -2H$. Furthermore, some of the surfaces satisfying that equation are the conoids of the 1st, 2nd, and 3rd kind exhibited in Examples 3.1–3.3.

Theorem 4.2. *Let M be a non-developable ruled surface with non-null base curve in a three-dimensional Minkowski space such that the condition $aH + bK$, $a \neq 0$, $b \in \mathbb{R}$, is a constant along each ruling. Then, M is an open part of one of the following surfaces*:

1. *the helicoid of the* 1*st kind as space-like or time-like surface,*
2. *the helicoid of the* 2*nd kind as space-like or time-like surface,*
3. *the helicoid of the* 3*rd kind as space-like or time-like surface,*
4. *the conjugate of Enneper's surfaces of the* 2*nd kind as space-like or time-like surface.*

Proof. In order to prove the theorem, we split it into two cases.

Case 1. As is described in Theorem 4.1 we assume that the non-developable ruled surface M of the three types M^1_+, M^3_+ or M^1_- is parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle\beta,\beta\rangle = \varepsilon_1(=\pm 1)$, $\langle\beta',\beta'\rangle = \varepsilon_2(=\pm 1)$ and $\langle\alpha',\beta'\rangle = 0$. Using the same notations given in Theorem 4.1 the Gaussian curvature K is given by

$$K = \langle N, N\rangle \frac{eg - f^2}{EG - F^2} = \frac{Q^2}{D^4}. \tag{4.14}$$

Differentiating K with respect to t we obtain

$$K_t = \frac{4\varepsilon_1 Q^2 t}{D^6}. \tag{4.15}$$

Suppose that the surface M satisfies the condition (1.3). First of all, we assume that $Q^2 - \varepsilon_1 t^2 > 0$. Then, by (1.3), (4.4) and (4.15) we can show that the coefficients of t^8, t^6, t^4, t^2 and t^0 are as follows:

$$t^8 : \varepsilon_1 a^2 J^2 = 0, \qquad t^6 : 2a^2 QJ(3F + QJ) - 4\varepsilon_1 a^2 Q'^2 + a^2 Q^2 J^2 = 0,$$
$$t^4 : 3\varepsilon_1 a^2 Q^2 (3F + QJ)(F + QJ) = 0,$$
$$t^2 : 3\varepsilon_1 a^2 Q^4 Q'^2 + a^2 Q^4 (3F + QJ)^2 - 64b^2 Q^4 = 0, \qquad t^0 : a^2 Q^6 Q'^2 = 0.$$

Thus we have $J = F = Q' = 0$ and $b = 0$. Since the coefficients of the powers of t consist of J, F, Q or Q', (4.2) implies that the mean curvature H is identically zero.

Next, we suppose that $Q^2 - \varepsilon_1 t^2 < 0$. In this case, by using (4.7) and (4.15) we can also show that the surface M is minimal. Consequently, by the proof of Theorem 4.1 the surface M is an open part of one of the helicoid of the 1st, 2nd and 3rd kind as space-like or time-like surface.

Case 2. Let M be the non-developable ruled surface of type M^2_+ or M^2_-. In this case, the curve α is space-like or time-like and β space-like but β' is light-like. We also use the notations given in Theorem 4.1. On the other hand, the Gaussian curvature K is obtained by

$$K = \frac{S^2}{q^2}, \tag{4.16}$$

and the differentiation of K with respect to t is given by

$$K_t = -\frac{4\varepsilon S^3}{q^3}. \tag{4.17}$$

Suppose that the surface M satisfies the condition (1.3). Then we obtain by (4.12) and (4.17)

$$2\varepsilon a^2 S(SS' + 2\varepsilon_1 S^2 R)^2 = 0,$$
$$a^2\varepsilon(SS' + 2\varepsilon S^2 R)(\varepsilon_1(SS' + 2\varepsilon_1 S^2 R) - 4S(\varepsilon_1 S' - SR)) = 0,$$
$$2a^2\varepsilon(\varepsilon_1 S' - SR)(S(\varepsilon_1 S' - SR) - \varepsilon_1(SS' + 2\varepsilon_1 S^2 R)) = 0,$$
$$a^2\varepsilon\varepsilon_1(\varepsilon_1 S' - SR)^2 - 64b^2 S^6 = 0,$$

which imply $S' = 0$, $R = 0$ and $b = 0$. Thus, the surface M is minimal by (4.9). Consequently, M is a conjugate of Enneper's surface of the 2nd kind as space-like or time-like surface according to Theorem 4.1. This completes the proof. □

Combining the results of Theorems 4.1, 4.2 and 2.1, we have the following theorem.

Theorem 4.3. *Let M be a non-developable ruled surface with non-null base curve in a three-dimensional Minkowski space. Then, the following are equivalent*:

1. *M has pointwise 1-type Gauss map.*
2. *M satisfies the equation $aK_{II} + bH = constant$, $a, b \in \mathbb{R} - \{0\}$, $2a - b \neq 0$, along each ruling.*
3. *M satisfies the equation $aH + bK = constant$, $a \neq 0$, $b \in \mathbb{R}$, along each ruling.*

Theorem 4.4. *Let $\alpha(s) + t\beta(s)$ be a non-developable ruled surface with non-null base curve in a three-dimensional Minkowski space such that the condition $aK_{II} + bK$, $a \neq 0$, $b \in \mathbb{R}$, is a constant along each ruling. Then, we have the following*:

1. *Non-cylindrical ruled surfaces such that $\beta'(s)$ is non-null are parts of one of the following surfaces*:
   (1) *the helicoid of the 1st kind as space-like or time-like surface,*
   (2) *the helicoid of the 2nd kind as space-like or time-like surface,*
   (3) *the helicoid of the 3rd kind as space-like or time-like surface.*
2. *Non-cylindrical ruled surfaces such that $\beta'(s)$ is null have vanishing second Gaussian curvature.*

Proof. In order to prove the theorem, we also split it into two cases.

Case 1. As is described in Theorem 4.1 we assume that the ruled surface M of the three types M_+^1, M_+^3 or M_-^1 is assumed to be parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s),$$

such that $\langle\beta, \beta\rangle = \varepsilon_1 (= \pm 1)$, $\langle\beta', \beta'\rangle = \varepsilon_2 (= \pm 1)$ and $\langle\alpha', \beta'\rangle = 0$. Likewise by Theorem 4.1 the second Gaussian curvature K_{II} and the Gaussian curvature K are given by (4.1) and (4.14), respectively. Suppose that the surface M satisfies the condition (1.4). First, we suppose that $Q^2 - \varepsilon_1 t^2 > 0$. Then, we have from (4.3) and (4.15)

$$t^{12} : \varepsilon_1 a^2 J^2 = 0, \qquad t^8 : 2\varepsilon_1 a^2 JQ^3(5F - QJ) - \varepsilon_1 a^2 Q^2 (F + 2QJ)(4JQ + F) = 0,$$
$$t^2 : 12\varepsilon_1 a^2 Q^8 Q'^2 + a^2 Q^8 (5F - QJ)^2 - 64 b^2 Q^8 = 0, \qquad t^0 : 4a^2 Q^{10} Q'^2 = 0,$$

which imply $J = F = Q' = 0$ and $b = 0$. Since the coefficients of the powers of t consist of J, F, Q or Q', the second Gaussian curvature K_{II} and the mean curvature H are identically zero by the help of (4.1) and (4.2). Thus, the surface M is minimal.

Next, we suppose that $Q^2 - \varepsilon_1 t^2 < 0$. In this case, by using (4.6) and (4.15) we can also show that M is minimal. Consequently, the surface M is an open part of one of the helicoids of the 1st, 2nd and 3rd kind as space-like or time-like surfaces depending on Case 1 of Theorem 4.1.

Case 2. Let M be a non-cylindrical ruled surface of type M_+^2 or M_-^2. In this case, the curve α is space-like or time-like and β space-like but β' is light-like. Suppose that the surface M satisfies the condition (1.4). Then we obtain by (4.11) and (4.17)

$$18\varepsilon a^2 SS'^2 = 0, \qquad 9\varepsilon\varepsilon_1 a^2 S'^2 - 64 b^2 S^6 = 0.$$

Therefore we have $S' = 0$ and $b = 0$. Thus, from (4.10) the second Gaussian curvature K_{II} is identically zero. This completes the proof. □

Finally, we investigate the relations between the second Gaussian curvature, the Gaussian curvature and the mean curvature of null scrolls in $\mathbb{E}_1^3$.

Theorem 4.5. *Let M be a null scroll in a three-dimensional Minkowski space. Then, M satisfies the equations $K = H^2$, $K_{II} = H^{-1}$.*

Proof. Let $\alpha = \alpha(s)$ be a light-like curve in $\mathbb{E}_1^3$ and $\beta = \beta(s)$ be a light-like vector field along α. Then, the null scroll M is parametrized by

$$x = x(s, t) = \alpha(s) + t\beta(s),$$

such that $\langle \alpha', \alpha' \rangle = 0$, $\langle \beta, \beta \rangle = 0$ and $\langle \alpha', \beta \rangle = 1$. Furthermore, without loss of generality, we may choose α as a null geodesic of M. We then have $\langle \alpha'(s), \beta'(s) \rangle = 0$ for all s. The induced Lorentz metric on M is given by $E = \langle \beta', \beta' \rangle t^2$, $F = 1$, $G = 0$ and the unit normal vector N is obtained by

$$N = \alpha' \times \beta + t\beta' \times \beta.$$

Thus, the component functions of the second fundamental form are given by

$$e = \langle \alpha'' + t\beta'', N \rangle, \qquad f = \langle \beta', \alpha' \times \beta \rangle = Q, \quad g = 0,$$

which imply $H = Q$ and $K = Q^2$.

If $\langle \beta', \beta' \rangle = 0$, then β' is either the zero vector or a null vector. If β' is the zero vector, the surface is flat because of $f = Q = 0$. Therefore, β' is a null vector and there is a non-zero smooth function ρ such that $\beta = \rho\beta'$. It is a contradiction by the properties of α and β.

Since it is described in Section 2, β' cannot be a time-like vector and thus we can choose the parameter s in such a way that $\langle \beta', \beta' \rangle = 1$. Let $\{\alpha', \beta, \beta'\}$ be a null frame in $\mathbb{E}_1^3$. Then, the vector β'' can be expressed by

$$\beta'' = -\alpha' + \langle \alpha', \beta'' \rangle \beta,$$

from which

$$e_{tt} = 2\langle \beta'', N_t \rangle = 2\langle \beta'', \beta' \times \beta \rangle = 2Q.$$

Therefore, using (1.1) and the above equations the second Gaussian curvature K_{II} is given by

$$K_{II} = \frac{1}{2Q^2} e_{tt} = \frac{1}{Q}.$$

Thus, it easily follows that $K_{II} = 1/H$ holds everywhere on a null scroll. This completes the proof. □

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